Vertical and Horizontal Asymptotes

(This handout is specific to *rational functions* $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions.)

What is an asymptote? An asymptote is a line that the graph of a function approaches.
IMPORTANT: The graph of a function may cross a horizontal asymptote any number of times, but the graph continues to approach the asymptote as the input increases and/or decreases without bound.

> ***IMPORTANT NOTE ON HOLES***: In order to find asymptotes, functions must ***FIRST*** be reduced. Factor both the numerator and denominator to see if they have any factors in common that can be "cancelled" out. Any factor that "cancels" produces a ***hole*** in the graph. A ***hole*** exists at a value of x if that value makes <u>both</u> the numerator and denominator equal to zero (0).

Example: $f(x)=\frac{x-2}{x^2-4} \Rightarrow \frac{\cancel{x-2}}{(\cancel{x-2})(x+2)} \Rightarrow \frac{1}{x+2}$ This means that the graph of $f(x)$ has a hole when $x=2$

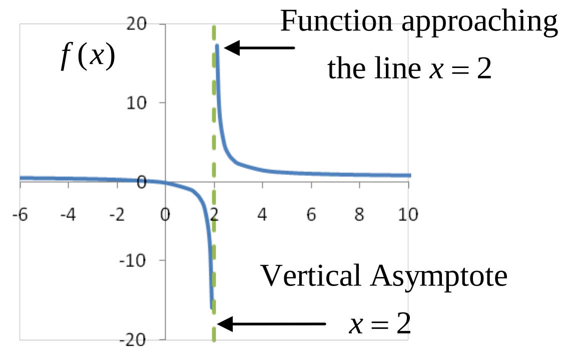
The *graphs* of $f(x)=\frac{x-2}{x^2-4}$ and $y=\frac{1}{x+2}$ are exactly the same except for the hole when $x=2$ on the first graph. Graphing calculators <u>*will not*</u> show the hole in the first graph, but the *table* will show an error when $x=2$.

Vertical Asymptotes: These vertical lines are written in the form: $x=k$, where k is a constant. Once a rational function is <u>*reduced*</u>, vertical asymptotes may be found by setting the denominator equal to zero (0) and solving for the input variable.

Example: $f(x)=\frac{2x+1}{3x-6}$

denominator $=0 \Rightarrow 3x-6=0 \Rightarrow 3x=6 \Rightarrow x=2$

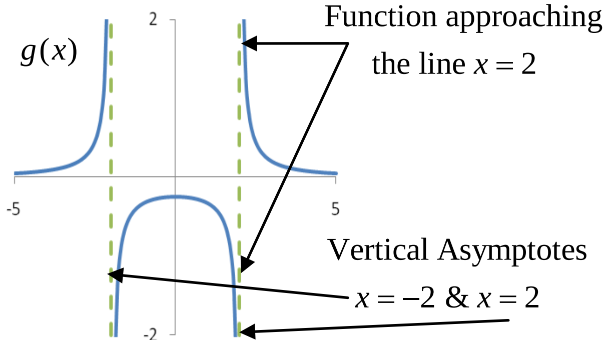
The graph of $f(x)$ has a the vertical asymptote $x=2$



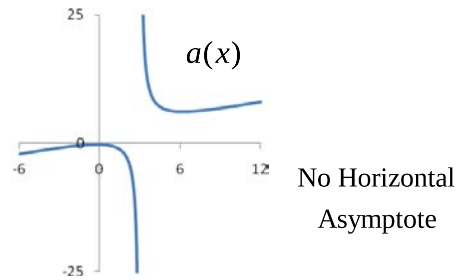
Example: $g(x)=\frac{1}{x^2-4}$

denominator $=0 \Rightarrow x^2-4=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$

The graph of $g(x)$ has the vertical asymptotes $x=-2$ and $x=2$



Horizontal Asymptotes: These horizontal lines are written in the form: $y = k$, where k is a constant. Because functions approach horizontal asymptotes for very large positive or negative input values, ONLY the terms with the *highest degree* (largest exponent) in both the numerator and denominator need to be considered when finding the horizontal asymptote. The resulting ratio (fraction) should then be reduced by "cancelling" common factors.

<table>
<tr><th>Case</th><th>Example</th><th>Graph</th></tr>
<tr>
<td>Case 1: If the result has no variables in the denominator, the function has no horizontal asymptote.</td>
<td>$a(x) = \dfrac{x^2+1}{x-3}$
$a(x) = \dfrac{x^2 + \cancel{1}}{2x - \cancel{6}} \Rightarrow \dfrac{x^2}{2x} = \dfrac{x}{2}$
No horizontal asymptote</td>
<td>
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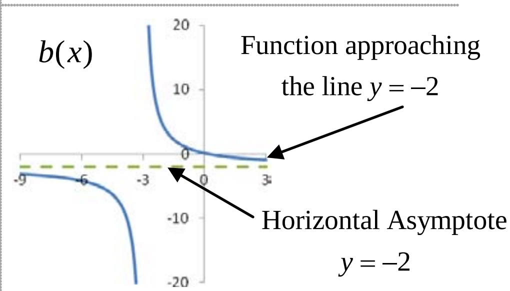
<td>Case 2: If the result is a number, the horizontal asymptote is $y =$ that number.</td>
<td>$b(x) = \dfrac{1-6x}{3x+7}$
$\dfrac{\cancel{1} - 6x}{3x + \cancel{9}} \Rightarrow \dfrac{-6x}{3x} = -2 \Rightarrow y = -2$
The horizontal asymptote is $y = -2$</td>
<td>
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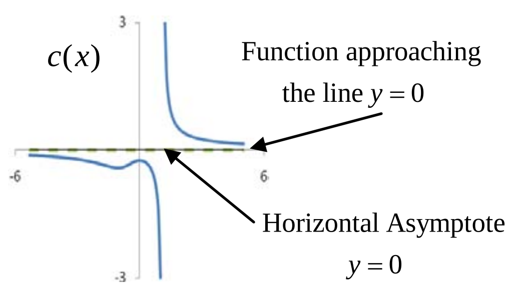
<tr>
<td>Case 3: If the result has no variables in the numerator, the horizontal asymptote is $y = 0$.</td>
<td>$c(x) = \dfrac{1+2x^2}{3x^3-4}$
$c(x) = \dfrac{\cancel{1} + 2x^2}{3x^3 - \cancel{4}} \Rightarrow \dfrac{2x^2}{3x^3} = \dfrac{2}{3x} \Rightarrow y = 0$
The horizontal asymptote is $y = 0$</td>
<td>
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Final Note:

There are other types of functions that have vertical and horizontal asymptotes not discussed in this handout.

There are other types of straight-line asymptotes called oblique or slant asymptotes.

There are other asymptotes that are not straight lines.

Example graphs of other functions with asymptotes: